QM Sept-2

Harmonic oscillator example

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \hbar\omega\left(a^\dagger a + 1/2\right)$$

Perturb: $V = \frac{1}{2} \varepsilon m\omega^2 x^2$

given: $V_{00} = \frac{1}{4} \varepsilon \hbar\omega$

$V_{20} = \frac{\varepsilon}{2\sqrt{2}} \hbar\omega = V_{02}$

others $V_{k0} = 0, \; k \neq 0, 2$

Find: a) 1^{st} & 2^{nd} order energy shift of ground state.

b) New ground state of the 1^{st} order

$$\Rightarrow \text{a)} \quad \Delta_0^{(1)} = \langle n^{(0)} | V | n^{(0)} \rangle$$

$$= \frac{c}{2} \varepsilon m\omega^2 \langle 0^{(0)} | (a^\dagger + a)(a^\dagger + a) | 0^{(0)} \rangle$$

$$= \frac{c}{2} \varepsilon m\omega^2 x^2 \langle 0^{(0)} | (a^\dagger + a)\left(| 1^{(0)} \rangle + 0 \right)$$

$$\Delta_0^{(1)} = \frac{c}{2} \varepsilon m\omega^2$$

$$\Delta_0^{(2)} = \sum_{m \neq n} \frac{|V_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$= \frac{|V_{20}|^2}{E_0^{(0)} - E_2^{(0)}} = -\frac{\varepsilon^2 \hbar\omega}{16}$$

$$|0^{(1)}\rangle = -\frac{\varepsilon}{4\sqrt{2}}\,|2^{(0)}\rangle$$

$\boxed{8)}$ c) Find the new E_0 exactly

Sakurai Example (5.1.5)

$$\Rightarrow\quad H = \frac{p^2}{2m} + \frac{1}{2}m\underbrace{(1+\varepsilon)\,\omega^2}_{\Omega^2}x^2$$

$$\Omega = \sqrt{1+\varepsilon}$$

$$E_0 = \frac{1}{2}\hbar\Omega = \frac{1}{2}\hbar\sqrt{1+\varepsilon}\;\omega$$

using the power series,

$$\sqrt{1+\varepsilon} = 1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \ldots$$

$$\Rightarrow\quad E_0 = \frac{1}{2}\hbar\omega + \frac{1}{2}\varepsilon\left(\frac{\hbar\omega}{2}\right) - \frac{1}{8}\varepsilon^2\left(\frac{\hbar\omega}{2}\right)$$

→ same as what we would get from perturbation theory. (approximation matches the exact solution only for this problem because perturbation was choosen that way.)

8) d) Normalize the approx ground state.

$$|0\rangle \approx |0^{(0)}\rangle - \frac{\varepsilon}{4\sqrt{2}}\,|2^{(0)}\rangle$$

$$\langle 0|0\rangle = 1 + \frac{\varepsilon^2}{32} \neq 1$$

- not normalized but not really far off (from being normalized) assume normalized state is:

$$|\tilde{0}\rangle = z^{1/2}|0\rangle$$

solving for z:

$$1 = \langle\tilde{0}|\tilde{0}\rangle = z\langle 0|0\rangle$$

$$\Rightarrow \quad z = \frac{1}{\langle 0|0\rangle} = \frac{1}{1+\frac{\varepsilon^2}{32}}$$

so,

$$|\tilde{0}\rangle = \frac{|0\rangle}{\sqrt{1+\frac{\varepsilon^2}{32}}}$$

$$= \frac{|0^{(0)}\rangle - \frac{\varepsilon}{4\sqrt{2}}|2^{(0)}\rangle}{1+\frac{\varepsilon^2}{64}} \qquad \left((1+x)^n = 1+nx+\ldots\right)$$

again using $(1+x)^n = 1+nx\ldots$, in denominator

?? *didn't really understand the discussion on the order of correction. Go through Sakurai to understand

$$|\tilde{0}\rangle = \left(1-\frac{\varepsilon^2}{64}\right)\left(|0^{(0)}\rangle - \frac{\varepsilon}{4\sqrt{2}}|2^{(0)}\rangle\right)$$

$$\langle n|n^{(0)}\rangle = 1$$

Normalization

$(H = H_0 + \lambda V)$

- we used: $\langle n^{(0)}|n\rangle \equiv 1$

$$\Rightarrow \langle n|n\rangle \neq 1$$

Normalize: $|\tilde{n}\rangle \equiv \sqrt{z_n}\,|n\rangle$; $z_n \in \mathbb{R}^+$

$$1 = \langle\tilde{n}|\tilde{n}\rangle = z_n\langle n|n\rangle$$

$\langle n^{(0)}|n^{(k)}\rangle = 0$ for $k>0$

the 'λ' term goes away then there's λ^2 term right after.

$$Z_n^{-1} = \langle n|n\rangle = \left(\langle n^{(0)}| + \lambda\langle n^{(1)}| + \dots\right)\left(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \dots\right)$$

$$= 1 + \lambda^2\langle n^{(1)}|n^{(1)}\rangle + O(\lambda^3)$$

$$Z_n^{-1} = 1 + \lambda \sum_{k\neq n}\left(\frac{\langle k^{(0)}|V_{kn}}{E_n^{(0)} - E_k^{(0)}}\right)\left(\sum_{l\neq n}\frac{|l^{(0)}\rangle V_{ln}}{E_n^{(0)} - E_l^{(0)}}\right)$$

since,

$$\langle k^{(0)}|l^{(0)}\rangle = \delta_{kl}$$

$$Z_n^{-1} = 1 + \lambda^2\sum_{k\neq n}\frac{|V_{kn}|^2}{\left(E_n^{(0)} - E_k^{(0)}\right)^2} + \dots$$

Probability Interpretation

$$\langle n^{(0)}|\tilde{n}\rangle = \langle n^{(0)}|\sqrt{Z_n}|n\rangle = \sqrt{Z_n}\,\underbrace{\langle n^{(0)}|n\rangle}_{1}$$

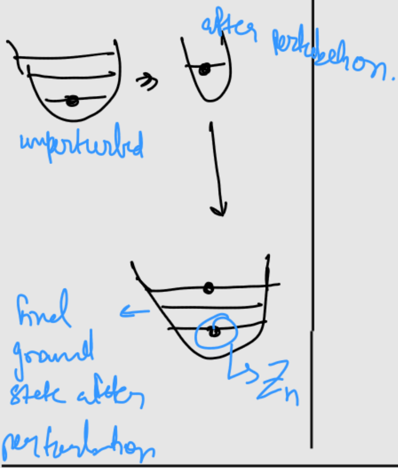
$$Z_n = \underbrace{|\langle n^{(0)}|\tilde{n}\rangle|^2}_{\text{interpretation}} = \text{probability}$$

interpretation

Prepare $|\tilde{n}\rangle$

measure H_0

$P(E_n^{(0)}) = Z_n$



in superposition of old ground state & new ground state