QM-2 Sep 18

Warm-up: Hydrogen atom

$$\vec{J} = \vec{L} + \vec{S}$$

a) Write $\vec{L}\cdot\vec{S}$ in terms of $\vec{J}^2$, $\vec{S}^2$, $\vec{L}^2$

b) Find $\langle n\, l\, s\, j\, m_j | \vec{L}\cdot\vec{S} | n\, l\, s\, j\, m_j \rangle$

$\Rightarrow$ a) from last lecture:

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

b) also from last class.

$$[V_{LS}, \vec{J}^2] = 0.$$

$$\Rightarrow \frac{1}{2}\langle n\, l\, s\, j\, m_j | \vec{J}^2 - \vec{L}^2 - \vec{S}^2 | n\, l\, s\, j\, m_j \rangle$$

$\vdots$

$$\frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

Midterm-1 $\rightarrow$ on 9/30 (Monday)

$\rightarrow$ upto homework-4

Today: spin-orbit coupling (5.3.2)
Zeeman Effect (5.3.3)

Spin Orbit Coupling

$$V_{LS} = \frac{1}{2m_e^2c^2} \frac{Ze^2}{4\pi\varepsilon_0 r^3} \vec{L}\cdot\vec{S}$$

Physical interpretation:

- Nucleus produces $\vec{E}(\vec{r})$
- electron velocity $\vec{v} = \vec{p}/m_e$
- electron's ref. frame: $\vec{B}' = \frac{1}{c^2}\vec{E}\times\vec{v}$

applied mag. field

$$V_{LS} \cong -\vec{\mu}\cdot\vec{B}' \propto \vec{S}\cdot(\vec{E}\times\vec{v}) \propto \vec{S}\cdot\underbrace{(\vec{r}\times\vec{p})}_{\vec{L}}$$

mag. moment

$$\propto \vec{L}\cdot\vec{S}$$

Rewrite as:

$$V_{LS} = \frac{1}{4} Z \alpha^4 m_e c^2 \left(\frac{a_0}{r}\right)^3 \frac{1}{\hbar^2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

- as discussed in the last class, we use coupled basis for this perturbation, because V_{LS} commutes with $\vec{L}^2, \vec{S}^2, \vec{J}^2, J_z$

↳ Example : commutation of $\vec{L}^2$ with V_{LS}

Use $[A,BC] = [A,B]C + B[A,C]$

$$[\vec{L}^2, \vec{L}\cdot\vec{S}] = [\vec{L}^2, L_i S_i] = \underbrace{[\vec{L}^2, L_i]}_{0} S_i + L_i \underbrace{[\vec{L}^2, S_i]}_{0}.$$

$$= 0$$

* we know V_{LS} is from $f(r)\,\vec{L}\cdot\vec{S}$.

$$[\vec{L}^2, V_{LS}] = [\vec{L}^2, f(r)\,\vec{L}\cdot\vec{S}]$$

$$= [\vec{L}^2, f(r)]\,\vec{L}\cdot\vec{S} + f(r)\,\underbrace{[\vec{L}^2, \vec{L}\cdot\vec{S}]}_{0}$$

(from previous example)

L is a f^n of θ and ϕ

$\therefore$ 0

The Last Question - Asimov

Q.E.D

going back to diagonalize V_{LS}:

$$\Rightarrow \langle n l' s' j' m_j' | V_{LS} | n l s j m_j \rangle \propto \delta_{l'l}\, \delta_{s's}\, \delta_{j'j}\, \delta_{m_j' m_j}$$

$l'l$ & $s's$ are always

equal for hydrogen

$\Rightarrow |n\, l\, s\, j\, m_j\rangle$ is the 0^{th} order basis.

* **Energy shift**

$$\Delta_{LS}^{(1)} = \langle n\, l j m_j | V_{LS} | n\, l j m_j \rangle \quad (s=1/2)$$

for $l=0$: expect "$\vec{L}\cdot\vec{S} = 0$"

check: $j = |l-s|, \ldots\ldots l+s = s = 1/2$

$$\vec{L}\cdot\vec{S}|n\, l\, s\, j\, m_j\rangle = \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]|n\, l\, j\, m_j\rangle$$

$$\Delta_{LS}^{(1)}(l=0) = \boxed{0}$$

for $l \neq 0$: use

$$\boxed{\left\langle \left(\frac{a_0}{r}\right)^3 \right\rangle = \frac{1}{l(l+\frac{1}{2})(l+1)}\left(\frac{Z}{n}\right)^3}$$

↳ proof in Sakurai using generating functions.

$$\Delta_{LS}^{(1)} = \frac{1}{4} Z \alpha^4 mc^2 \langle n\, l\, j\, m_j | \left(\frac{a_0}{r}\right)^3 \left(\frac{\vec{J}^2 - \vec{L}^2 - \vec{S}^2}{\hbar^2}\right) | n\, l\, j\, m_j \rangle$$

$$\Delta_{LS}^{(1)} = \frac{1}{4} Z^4 \alpha^4 mc^2 \frac{[j(j+1) - l(l+1) - s(s+1)]}{n^3\, l(l+1/2)(l+1)}$$

substitution

let $j = l + \frac{1}{2}\sigma \;, \sigma = \pm 1$

$$= \frac{1}{4} (Z\alpha)^4 m_e c^2 \frac{\sigma}{(l+1/2)(j+1/2)\, n^3}$$

Total shift

- $l \neq 0$: $\underbrace{\Delta_{Rel}^{(1)}}_{\text{relativistic correction}} = \Delta_{KE}^{(1)} + \Delta_{LS}^{(1)} + \cancel{\Delta_{Darwin}^{(1)}}$ 0 (for $l \neq 0$)

$$\Delta_{Rel}^{(1)} = \frac{1}{2} \frac{(Z\alpha)^4 m_e c^2}{n^3} \left(\frac{3}{4n} - \frac{1}{j+1/2} \right)$$

Proof is a Problem on PS4

- $l = 0$: $\Delta_{rel}^{(1)} = \Delta_{KE}^{(1)} + \cancel{\Delta_{LS}^{(1)}} + \Delta_{Darwin}^{(1)}$

0 (for $l = 0$)

$\boxed{\text{gives the same result as } l \neq 0!}$

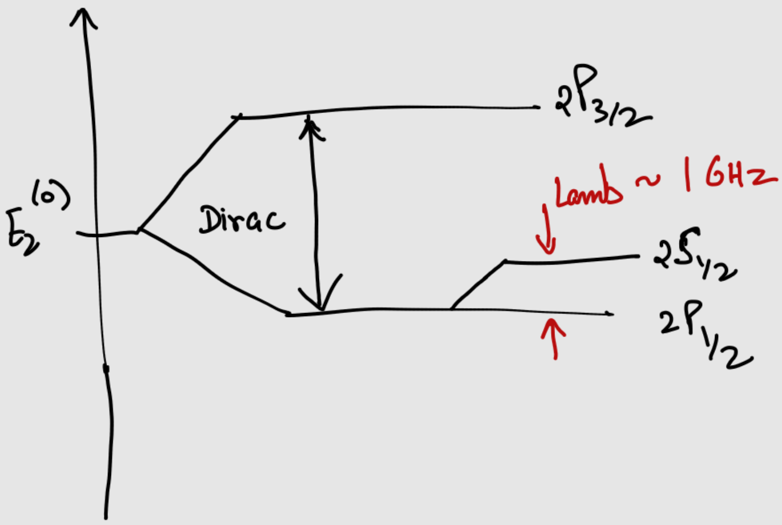
- energies are still degenerate in l, m_j

m_j → expected

l → not expected

(← broken by lamb shift at order α^5

- Lamb shift comes from Q.E.D)



have to calculate on (PS4)

Hint: ← energy shift drawn some-what to scale