QM-2 Sep 16

Review: Addition of angular momentum

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

coupled basis $|j_1 j_2 j m\rangle$

$$\vec{J}^2 |j_1 j_2 j m\rangle = \hbar^2 j(j+1) |j_1 j_2 j m\rangle$$

$j = ?$

$\Rightarrow \quad j = |j_1 - j_2|, \underbrace{\ldots\ldots}_{\text{steps of } 1} j_1 + j_2$

2) Hydrogen atom

consider $n=2$

let $\vec{J} = \vec{L} + \vec{S}$

where $\vec{S}$ = electron spin

- list allowed values of l & j

$\Rightarrow \quad n=2, \quad l = 0, 1$

electron $l = -\frac{1}{2}, \frac{1}{2}$

$j = |j_1 - j_2|, \ldots\ldots |j_1 + j_2|$

$= \frac{1}{2}, \frac{3}{2}$

l	s	j	
0	$1/2$	$1/2$	$^2S_{1/2}$
1	$1/2$	$1/2$	$^2P_{1/2}$
1	$1/2$	$1/2$	$^2P_{3/2}$

didn't consider $-1/2$ b/c this is the angular-momentum no., $-1/2$ is for spin quantum no. m_s.

Spectroscopic notation

$$^{2s+1}[\,l\,]_j$$

represented by letters,

S : 0
P : 1
d : 2
f : 3
g : 4
h : 5
⋮ : ⋮

(5.3.2) Fine structure of H spin-orbit coupling

not part of curriculum, discussed for fun.

Dirac Equation (8.3)

- Relativity + QM: Find Hamiltonian w/ eigenval $E = \sqrt{(pc)^2 + (mc^2)^2}$ for free particle

solution: $H = \vec{\alpha} \cdot \vec{p} + \beta m$ (c=1 units)
$(= \vec{\alpha} \cdot \vec{p} + \beta mc^2)$

Need:

$$\vec{p}^{\,2} + m^2 = H^2 = (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)$$

$$= \underbrace{\alpha_i \alpha_j}_{\delta_{ij}} p_i p_j + \underbrace{(\alpha_i \beta + \beta \alpha_i)}_{0} p_i m + \underbrace{\beta^2}_{1} m^2$$

on comparing L.H.S and R.H.S $\Longrightarrow$

Solution:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \Rightarrow \alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix},$$

$$\alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix},$$

$$\alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

σ_i = Pauli Matrices

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$\Rightarrow$ Spin $1/2$, but twice

- matter, antimatter

- correct for electron

E&M field: $\vec{p} \to \vec{p} - q\vec{A}$ Vector potential

$H \to H + q\phi$ Scalar potential

$$H = \vec{\alpha}\cdot(\vec{p} - q\vec{A}) + \beta m + q\phi$$

$$H\Psi = E\Psi \quad ; \quad \vec{p} = -i\hbar\vec{\nabla}$$

* Reduce Dirac → Schrödinger + Perturbations.

Assume $|E - mc^2| \ll mc^2$

Result.

$$H \approx H_0 + V_{KE} + V_{LS} + V_{Darwin}$$

spin-orbit coupling

more in next problem set.

$$\boxed{V_{LS} = \frac{1}{2m_e^2c^2}\,\frac{q}{r}\,\frac{d\phi}{dr}\,\vec{L}\cdot\vec{S}}$$

$$\begin{cases} \phi(r) = \dfrac{Ze}{4\pi\varepsilon_0 r} \\ \dfrac{d\phi}{dr} = \dfrac{-Ze}{4\pi\varepsilon_0 r^2} \end{cases}$$

$q = -e$

for hydrogen:

$$V_{LS} = \frac{1}{2m_e^2 c^2} \frac{Ze^2}{4\pi\varepsilon_0 r^3} \vec{L}\cdot\vec{S}$$

→ this equation tells us that

– include spin $|n\, l\, m_l\, m_s\rangle$ "decoupled basis"

$|n\, l\, j\, m_j\rangle$ "coupled basis"

* Correct basis for degenerate Perturbation theo.

$$\vec{J} = \vec{L} + \vec{S}$$

$$\vec{J}^2 = (\vec{L}+\vec{S})^2 = \vec{L}^2 + 2\vec{L}\cdot\vec{S} + \vec{S}^2$$

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\underbrace{\vec{J}^2 - \vec{L}^2 - \vec{S}^2}\right)$$

↳ all present in coupled basis, so it is the right choice of basis here.

$$[V_{LS}, J^2] = 0$$

$$\langle n\, l'\, j'\, s'\, m_j' | V_{LS} | n\, l\, j\, s\, m_j\rangle \propto \delta_{ll'}\, \delta_{jj'}\, \delta_{ss'}\, \delta_{m_j m_j'}$$

⇒ diagonal in coupled basis