QM-2 Sep-11

Today : Finish Linear Stark (5.2.1)
Relativistic K.E. (5.3.1)

Warm Up:

$$V = \begin{pmatrix} 3 & 4 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

a) Show that $\lambda = 7$ is an eigen value.

b) Find eigenvec w/ $\lambda = 7$

⇒ a) $|V - \lambda I| = \begin{vmatrix} 3-\lambda & 4 & 0 \\ 4 & 6-\lambda & 0 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$

$$(3-\lambda)(6-\lambda)(7-\lambda) - 4[(7-\lambda)4 - 0]$$

$$+0$$

$$(7-\lambda)\,[\quad - \quad] = 0$$

⇒ $\lambda = 7$ is an eigen value

b) $$\begin{pmatrix} 3 & 4 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 7 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$3a_1 + 4a_2 = 7a_1$$

$$4a_1 + 6a_2 = 7a_2$$

$$7a_3 = 7a_3 \Rightarrow a_3 = 1$$ (1st non trivial eigenvector b/c others two are zero)

$$\Rightarrow \quad -4a_1 + 4a_2 = 0$$

$$4a_1 - a_2 = 0$$

$$3a_2 = 0$$

$$a_2 = 0$$

$$\Rightarrow a_1 = a_2 = 0$$

$$\therefore \boxed{|a\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

Block Diagonal matrix :

$$V = \begin{pmatrix} \boxed{\begin{matrix} 3 & 4 \\ 4 & 6 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{7} \end{pmatrix}$$

"block diagonal matrix"

$$0 \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} = 7 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$0 \qquad 0$$

(can have an intuition about eigenvectors of element in matrix which is a block)

LAST Time :

effect $\vec{E}$ on atom (stark effect)

Review : $H_0 = \dfrac{\vec{p}^{\,2}}{2m_e} - \dfrac{Ze^2}{4\pi\varepsilon_0 r}$ ($Z=1 \rightarrow$ hydrogen)

$$V_{stark} = e\,\mathcal{E}\,z$$

- unperturbed solution $|n\,l\,m\rangle$

$$\psi_{nlm}(\vec{r}) = R_{nl}(r)\; Y_l^m(\theta, \phi)$$

$$E^{(0)}_{nlm} = \frac{\hbar^2}{2m_e a_0^2}\,\frac{1}{n^2} = \frac{-13.6\,eV}{n^2}$$

Perturb $n=2$ states:

$$\left.\begin{aligned} |m_1^{(0)}\rangle &= |2\ 0\ 0\rangle \\ |m_2^{(0)}\rangle &= |2\ 1\ 0\rangle \\ |m_3^{(0)}\rangle &= |2\ 1\ 1\rangle \\ |m_4^{(0)}\rangle &= |2\ 1\ {-1}\rangle \end{aligned}\right\} \rightarrow [V]_{n=2} = -3e\,\mathcal{E}a_0 \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{eigen vals } \mu}$$

eigenvalues: $\Delta^{(1)}_{n=2} = \pm 3e\,\mathcal{E}a_0,\ 0,\ 0$

*read notes to see if there's material on block matrices & finding eigen-values easily

$$[V]_{n=2} = -3e\,\mathcal{E}a_0 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

→ blocks (⇒ two zeros are the eigen-values of matrix)

Eigenvecs:

$$|l_i^{(0)}\rangle = \sum_j a_{ij}\,|m_j^{(0)}\rangle$$

$$\Delta^{(1)}_{21} = 3e\,\mathcal{E}a_0 \qquad (\mu=-1)$$

$n=2$, i

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$i=1$

$$a_{11} + a_{12} = 0$$

$$a_{13} = 0, \quad a_{14} = 0$$

$$\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$$

$$|l_1^{(0)}\rangle = \frac{1}{\sqrt{2}}|2\ 0\ 0\rangle - \frac{1}{\sqrt{2}}|2\ 1\ 0\rangle$$

* <u>solving for the same using tricks of box diagonal matrices</u>

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu = -1$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$a_{11} + a_{12} = 0 \Rightarrow a_{11} = \frac{1}{\sqrt{2}} = -a_{12}$$

$$\Rightarrow \boxed{|\ell_1^{(0)}\rangle = \frac{1}{\sqrt{2}}|2\,0\,0\rangle - \frac{1}{\sqrt{2}}|2\,1\,0\rangle}$$

which is same as above.

- The same process can be used to obtain other perturbed eigenvectors:

$$\mu = 1, \quad \Delta_{22}^{(1)} = -3e\mathcal{E}a_0$$

$$\mu = 0, \quad \Delta_{23}^{(1)} = 0 = \Delta_{24}^{(1)}$$

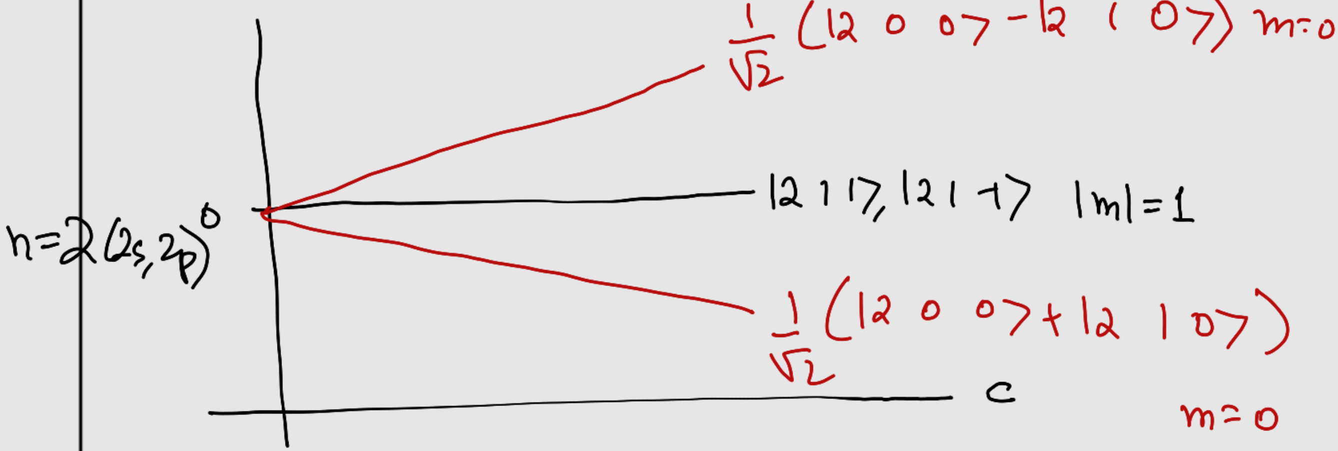
$$\vec{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{a}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

→ we know these are eigenvectors from the block diagonal. any linear combination of these is also an eigenvector.

$$\boxed{|\ell_2^{(0)}\rangle = \frac{1}{\sqrt{2}}|2\,0\,0\rangle + \frac{1}{\sqrt{2}}|2\,1\,0\rangle}$$

$$\boxed{|\ell_3^{(0)}\rangle = |2\,1\,1\rangle}$$

$$\boxed{|\ell_4^{(0)}\rangle = |2\,1\,-1\rangle}$$



Relativistic K.E. (5.3.1)

Relativity:

$$KE = \sqrt{(\vec{p}c)^2 + (m_e c^2)^2} - m_e c^2$$

$$= m_e c^2 \sqrt{1 + \frac{\vec{p}^2}{(m_e c)^2}} - m_e c^2$$

using the bionomial expension

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$$

$$\approx m_e c^2 \left(1 + \frac{1}{2}\left(\frac{p^2}{m^2 c^2}\right) - \frac{1}{8}\frac{(p^2)^2}{m^4 c^4} + \ldots\right) - m_e c^2$$

$$= \frac{\vec{p}^2}{2m_e} - \underbrace{\frac{1}{8}\frac{(\vec{p}^2)^2}{m_e^3 c^2}}_{V_{KE}}$$

(kinetic energy perturbation)

Rotational Symmetry

$$[V_{KE}, L_\alpha] = 0 \qquad \alpha = x, y, z$$

$$\Rightarrow \quad [V_{KE}, \vec{L}^2] = 0$$

$$\Rightarrow \quad [V_{KE}, L_z] = 0$$

Since,

$$\langle n' l' m' | V | n\, l m \rangle = 0 \quad \text{if} \quad l \neq l'$$

$$\Rightarrow \langle n' l' m' | V_{KE} | n\, l m \rangle = \delta_{l'l} \delta_{m'm} \langle n' l m | V_{KE} | n\, l m \rangle$$

$$\Rightarrow (V_{KE})_n \quad \text{is diagonal}$$

$\Rightarrow |n\, l\, m\rangle$ are correct zeroth order kets.

$$\Delta^{(1)}_{nlm} = \langle n\, l m | V_{KE} | n\, l\, m \rangle = \ldots\ldots$$

SEE sakurai for clever trick to calculate these.

$$= (Z\alpha)^2 \frac{E_n^{(0)}}{n^2} \left(-\frac{3}{4} + \frac{n}{l + \frac{1}{2}} \right)$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{\hbar c} \approx \frac{1}{137}$$