Oct - 9

Today : T.D.P.T

$\Rightarrow$ { (5.5.2, 5.7.2) } Read

Recall : $H(t) = H_0 + V(t)$

$$U(t,t_0) = U^{(0)}(t,t_0) + U^{(1)}(t,t_0) + \ldots$$

$$U^{(0)}(t,t_0) = U_0(t-t_0) = e^{-iH_0(t-t_0)/\hbar}$$

$$U^{(1)}(t,t_0) = \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, U_0(t-t_1)V(t_1)U_0(t_1-t_0)$$

WARM-UP :

$$H_0 = \hbar\omega(a^\dagger a + 1/2)$$

$$E_n = \hbar\omega(n+1/2) = \hbar\omega_n$$

$$\omega_n = \omega(n+1/2)$$

$$V(t) = g(a+a^\dagger)$$

initial condition :

$$|\alpha, t_0 = 0\rangle = |0\rangle$$

then : $|\alpha, t\rangle = [U^{(0)}(t,0) + U^{(1)}(t,0) + \ldots]\,|0\rangle$

a) Find $U^{(0)}(t,0)|0\rangle$

b) Find $U^{(1)}(t,0)|0\rangle$

$\Rightarrow$ (a) $U^{(0)}(t,0)|0\rangle = e^{-iE_0 t/\hbar}|0\rangle$

$\rightarrow E_0 = \hbar\omega_0 = \frac{\hbar\omega}{2}$

$$= e^{-\frac{i\omega t}{2}}|0\rangle$$

(b) $U^{(1)}(t,t_0) = \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, U_0(t-t_1) V(t_1) U_0(t_1-t_0)$

$$V(t) = g(a+a^{+})$$

E₀ for both b.c. hamiltonian is independent of time.

$$U_0(t_1-t_0) = e^{-\frac{iE_0 t_1}{\hbar} - \left(-\frac{iE_0 t_0}{\hbar}\right)}$$

$$= e^{-\frac{iE_0}{\hbar}(t_1-t_0)}$$

$$\therefore U_0(t_1,t_0) = e^{-i\omega t}$$

$$U(t,t_1) = U(t-t_1) = e^{-i\frac{E_0(t-t_1)}{\hbar}}$$

$$U^{(1)}(t,t_0) = \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, e^{-i\frac{E_0(t-t_1)}{\hbar}}\, g(a+a^{+})\, e^{\frac{-iE_0(t_1-t_0)}{\hbar}}$$

at t_1 $E = E_1$? how? (from interaction picture?)

integral

$$\therefore U^{(1)}(t,t_0) = I\, g(a+a^{+})$$

* in T.D.P.T, the choice of basis kets are usually eigenstates of unperturbed hamiltonian at $t=0$. These states are then evolved with time.

(C) Let $|1, t^{(0)}\rangle = U^{(0)}(t,0)|1\rangle$

$$= e^{-iH_0 t/\hbar}|1\rangle$$

$$= e^{-iE_1 t/\hbar}|1\rangle$$

Find $\langle 1, t^{(0)}|U^{(1)}(t,0)|0\rangle$

from b)

$$U^{(1)}(t,t_0)|0\rangle = I \quad g(a+a^{\dagger})|0\rangle$$

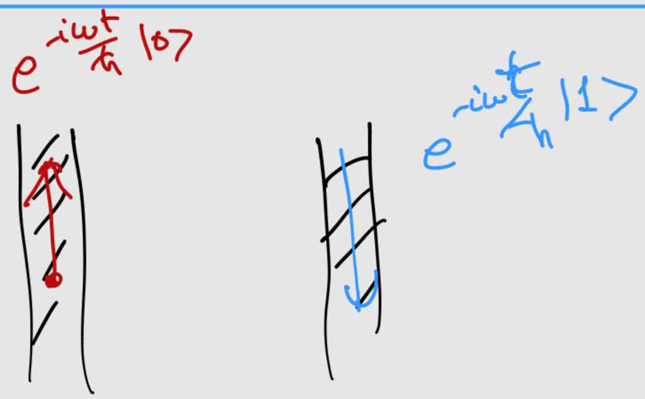
$$= I g|1\rangle$$

* probability amplitudes

$$\Rightarrow \langle 1, t^{(0)}|Ig|1\rangle$$

⋮

$$-\frac{g}{\hbar\omega}(e^{i\omega t}-1)$$



walkways at airport
top-down view

d) Find Prob. of being in $|1\rangle$ at time t.

$$\Rightarrow \quad P_1(t) = \left|\langle 1|\alpha, t\rangle\right|^2 = \left|\langle 1|e^{i\omega_1 t}|\alpha, t\rangle\right|^2$$

where $\omega_1 = \dfrac{E_1}{\hbar}$

$$= \left|\langle 1, t^{(0)}|\alpha, t\rangle\right|^2 = \left|\langle 1, t^{(0)}|\underbrace{U(t,0)}_{U^{(1)}}|\alpha, 0\rangle\right|^2$$

$$\simeq \left|\langle 1, t^{(0)}|U^{(1)}(t,0)|0\rangle\right|^2$$

using the result of part (c)

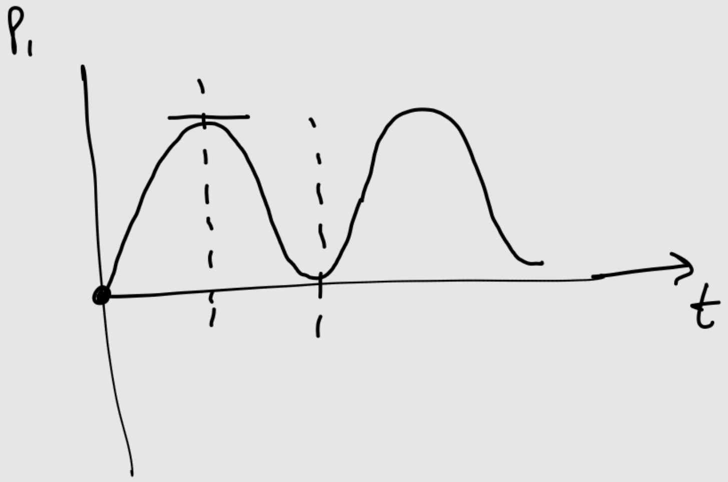
$$= \frac{g^2}{(\hbar\omega)^2}\left|e^{i\omega t} - 1\right|^2$$

$$= \left(\frac{g}{\hbar\omega}\right)^2 \left|e^{i\omega t/2}\right|^2 \left|e^{i\omega t/2} - e^{-i\omega t/2}\right|^2$$

$$= \boxed{4\underbrace{\left(\frac{g}{\hbar\omega}\right)^2}_{\ll 1}\sin^2(\omega t/2)}$$

prob
$0 \xrightarrow{t} 1$

b/c perturbation should be very small so the factor g must be very small (smaller than $\hbar\omega$, the gap b/w energy levels)



<u>Generalizing this process:</u>

$$H(t) = H_0 + V(t)$$

$\Rightarrow$ $H_0|n\rangle = E_n|n\rangle = \hbar\omega_n|n\rangle$

- preferred basis: $e^{-iH_0t/\hbar}|n\rangle = |n, t^{(0)}\rangle$
$= e^{-i\omega_n t}|n\rangle$

→ interaction picture

$$|n, t^{(0)}\rangle_I = e^{iH_0t/\hbar}|n, t^{(0)}\rangle = |n\rangle$$

simpler in interaction pic.

Following this discussion on basis in Schrödinger picture and interaction picture:

initial condition: (in Schrödinger pict.) $|\alpha, t_0\rangle = |i, t_0^{(0)}\rangle$ i^{th} state (i for initial)

$= e^{-iH_0t_0}|i\rangle = e^{-i\omega_i t}$

initial condition in interaction picture.

$$|\alpha, t_0\rangle_I = |i\rangle$$

- First-order transition amplitude:

Schrödinger picture :

$$\langle n, t^{(0)} | U^{(1)}(t, t_0) | i, t_0^{(0)} \rangle$$

$$= \langle n | \underbrace{e^{iH_0 t/\hbar} \; U^{(1)}(t, t_0) \, e^{-iH_0 t_0/\hbar}}_{U_I^{(1)}(t, t_0)} | i \rangle$$

interaction picture :

$$= \langle n | U_I^{(1)}(t, t_0) | i \rangle$$

$$\langle n | U_I^{(1)}(t, t_0) | i \rangle = \langle n | \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 \, V_I(t_1) | i \rangle$$

$$= \langle n | \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 \, e^{iH_0 t_1/\hbar} \, V(t_1) \, e^{-iH_0 t/\hbar} | i \rangle$$

$$= \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 \, \langle n | e^{i\omega_n t_1} \, V(t_1) \, e^{-i\omega_i t_1} | i \rangle$$

$$\boxed{= \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 \, e^{i(\omega_n - \omega_i) t_1} \, V_{ni}(t_1)}$$

$$V_{ni}(t) = \langle n | V(t) | i \rangle$$

$$\omega_n = E_n / t$$

$$\omega_i = E_i / \hbar$$