Oct 30

Midterm-2 on Nov 6, Wednesday

↳ T.D.P.T, FGR, dipole transitions

- 1 page notes.

Today: Review Scattering

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Warm-up:

Given:

T.D.S.E $\quad i\hbar \frac{d}{dt}|\psi(t)\rangle = (H_0 + V(t))|\psi(t)\rangle$

Interaction picture:

$$|\psi_I(t)\rangle = e^{iH_0 t/\hbar}|\psi(t)\rangle$$

* clear up the interaction picture & Schrodinger picture quantities

Find: $\quad i\hbar \frac{d}{dt}|\psi_I(t)\rangle = \underline{\quad ? \quad} |\psi_I(t)\rangle$

$$\Rightarrow LHS = i\hbar \frac{d}{dt}|\psi_I(t)\rangle$$

$$= i\hbar \frac{d}{dt}\left(e^{iH_0 t/\hbar} |\psi(t)\rangle\right)$$

$$= i\hbar \left[e^{iH_0 t/\hbar} \frac{d}{dt}\left(|\psi(t)\rangle\right) + |\psi(t)\rangle \frac{iH_0}{\hbar} e^{\frac{iH_0 t}{\hbar}} \right]$$

$$= e^{iH_0 t/\hbar} \underbrace{i\hbar \frac{d}{dt}|\psi(t)\rangle} - H_0 \, e^{iH_0 t/\hbar} |\psi(t)\rangle$$

$\hookrightarrow H_0 + V(t) \, |\psi(t)\rangle$

(given T.D.S.E)

$$= e^{iH_0 t/\hbar} \left(H_0 + V(t)\right) |\psi(t)\rangle - H_0 \, e^{iH_0 t/\hbar} |\psi(t)\rangle$$

$$= \cancel{e^{iH_0 t/\hbar} H_0} |\psi(t)\rangle + e^{iH_0 t/\hbar} V(t) |\psi(t)\rangle$$

$$- \cancel{H_0 e^{iH_0 t/\hbar}} |\psi(t)\rangle$$

$$= \underbrace{e^{iH_0 t/\hbar} V(t) \, e^{-iH_0 t/\hbar}}_{V_I(t)} \cdot \underbrace{e^{iH_0 t/\hbar} |\psi(t)\rangle}_{|\psi_I(t)\rangle}$$

$$\boxed{i\hbar \frac{d}{dt}|\psi_I(t)\rangle = V_I(t) \, |\psi_I(t)\rangle}$$

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Solve :

$$\frac{d}{dt}\,|\psi_I(t)\rangle = \frac{1}{i\hbar}\, V_I(t)\,|\psi_I(t)\rangle$$

integrate.

$$\int_{t_0}^{t} dt'\,\frac{d}{dt'}\,|\psi_I(t')\rangle = \underbrace{|\psi_I(t)\rangle - |\psi_I(t_0)\rangle}_{\text{fundamental thm of calculus}} \rightarrow \text{LHS integral}$$

$$= \frac{1}{i\hbar}\int_{t_0}^{t} V_I(t')\,|\psi_I(t')\rangle\, dt'$$

integral of RHS.

$$|\psi_I(t)\rangle = |\psi_I(t_0)\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt'\, V_I(t')\,|\psi_I(t')\rangle$$

$$\left(= |\psi_I(t_0)\rangle + \frac{1}{i\hbar}\int \ldots \right)$$

$O(V)$

$$|\psi_I(t)\rangle = |\psi_I(t_0)\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt'\, V_I(t')\,|\psi_I(t_0)\rangle$$

$$+\left(\frac{1}{i\hbar}\right)^2 \int_{t_1}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \; V_I(t_2) V_I(t_1) |\psi_I(t_0)\rangle$$

$$+ \dots$$

$$\equiv U_I(t, t_0) |\psi_I(t_0)\rangle$$

$$U_I(t, t_0) = \mathbb{1} + \frac{1}{i\hbar} \int_{t_0}^{t} dt' \, V_I(t') + \dots$$

- Initial state : $|\psi_I(t_0)\rangle = |i\rangle$

$$H_0 |i\rangle = \hbar \omega_i |i\rangle$$

$$|\psi_I(t)\rangle = |i\rangle + \frac{1}{i\hbar} \int_{t_0}^{t} dt' \, V_I(t') + \dots$$

$$\langle n|\psi_I(t)\rangle = \delta_{ni} + \frac{1}{i\hbar} \int_{t_0}^{t} dt' \, \langle n| V_I(t') |i\rangle$$

$$= \delta_{ni} + \frac{1}{i\hbar} \int_{t_0}^{t} dt' \, \langle n| e^{iH_0 t'/\hbar} \, V(t') \, e^{-iH_0 t'/\hbar} |i\rangle$$

$$= \delta_{ni} + \frac{1}{i\hbar}\int_{t_0}^{t} dt' \, e^{i(\omega_n - \omega_i)t} \langle n|V(t)|i\rangle + \ldots$$

$$c_n^{(1)} = \frac{1}{i\hbar}\int_{t_0}^{t} dt' \, e^{i\omega_{ni}t'} V_{ni}(t')$$

$$|\psi_I(t)\rangle = \sum_n |n\rangle \underbrace{\langle n|\psi_I(t)\rangle}_{c_n(t)} = \sum_n c_n(t)|n\rangle$$

$$c_n(t) = c_n^{(0)} + c_n^{(1)}(t) + \ldots$$

$$|\psi(t)\rangle = e^{-iH_0t/\hbar}|\psi_I(t)\rangle = \sum_n c_n(t)e^{-i\omega_n t}|n\rangle$$

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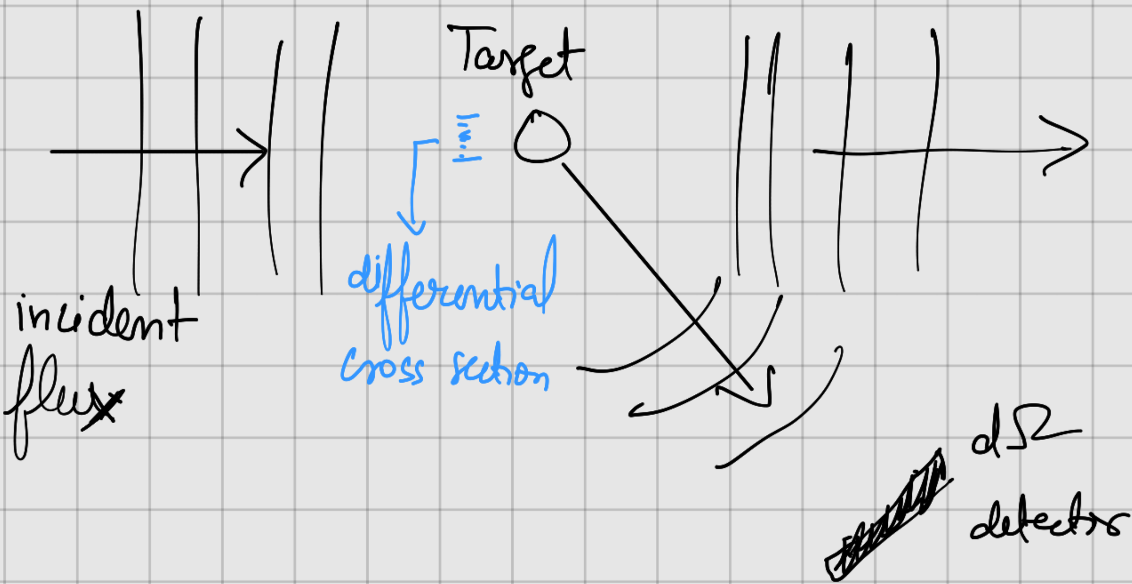
Scattering Theory

- Particle in 3D (wave), no spin
- target (infinitely massive) at $\vec{r} = 0$

same theory works for two particles with recoil in center of mass frame.



- incident flux $F = \dfrac{\#\ \text{particles}}{\text{Area} \times \text{time}}$

rate of detection

- detect $\dfrac{dR}{d\Omega} = \dfrac{\#\ \text{particles}}{\text{time} \times d\Omega}$

- <u>cross-section</u> :

$$\frac{dR}{d\Omega} = F \underbrace{\left(\frac{d\sigma}{d\Omega}\right)}$$

↳ differential cross-section.
(essentially the "area" of the target)

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{F}\frac{dR}{d\Omega}$$

→ rate can be calculated using Fermi's golden rule.

FGR gives the transition rates.

$$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r}) = H_0 + V$$

$$H_0 = \frac{\vec{p}}{2m} \quad \text{(free particle)}$$

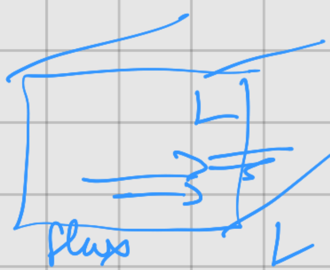
- use periodic boundary conditions in box of dimension $L \times L \times L$

$$\varphi_{\vec{k}}(\vec{r}) = \frac{1}{\sqrt{L^3}} e^{i\vec{k}\cdot\vec{r}}$$

$$\vec{k} \simeq \frac{2\pi}{L}(n_x, n_y, n_z)$$

- initial state: $\varphi_i(\vec{r}) = \dfrac{1}{\sqrt{L^3}} e^{i\vec{k}_i\cdot\vec{r}}$

wave vectors pointing toward the target



- flux: $F = \dfrac{1 \text{ particle}}{\underbrace{L^2}_{\text{area}} \cdot \underbrace{\frac{L}{v}}_{\text{time } \left(\frac{\text{dist.}}{\text{speed}}\right)}}$

$$= \frac{v}{L^3} = \frac{\hbar k_i / m}{L^3}$$

* final state would have a wave vector pointing towards the detector, so that we can use F.G.R to calculate the transition b/w these initial and final states.

F.G.R rate into $d\Omega$

$$dR = \frac{2\pi}{\hbar} \left| \langle k_f | V | k_i \rangle \right|^2 \rho_f (E_i)$$

$$\rho_f = \frac{m L^3 k}{8 \pi^3 \hbar^2} d\Omega \quad \left(|\vec{k_i}| = |\vec{k_f}| = k \right)$$

$$\frac{dR}{d\Omega} = \frac{2\pi}{\hbar} |V_{ki}|^2 \frac{m L^3 k}{8\pi^3 \hbar^2} = \frac{m L^3 k |V_{fi}|^2}{(2\pi)^2 \hbar^3}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{F} \frac{dR}{d\Omega} = \frac{L^3 m}{\hbar k} \frac{m L^3 k}{(2\pi)^2 \hbar^3} \left| \int d^3 r \frac{1}{L^3} e^{-i \vec{k_f} \cdot \vec{r}} V(\vec{r}) e^{i \vec{k_i} \cdot \vec{r}} \right|^2$$

Born Approximation

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi \hbar^2} \right) \left| \int d^3 \vec{r} \, e^{i (\vec{k_i} - \vec{k_f}) \cdot \vec{r}} V(\vec{r}) \right|^2}$$

* gives differential cross section for every input potential $V(\vec{r})$

1^{st} order interaction
└──→ captured by Born approximation.

higher order interaction
└──→ ↳ born approximation fails

* ⇒ for Bohr approximation to be valid, interaction order (perturbation potential) can't be high.