QM-2
Oct-16

Today: Sakurai 5.7.3
- constant Perturbation
- transition to continuum.
- fermi's Golden Rule

Warm-up:

$$H(t) = H_0 + V(t)$$

$$H_0|n\rangle = E_n|n\rangle$$

basically harmonic oscillator (energy level separation similar to harmonic oscillator)

$$E_n = \hbar\Delta n \quad , n = 0, 1, 2, \ldots$$

$$V(t) = \begin{cases} 0 & , t < 0 \\ V & , t \geq 0 \end{cases}$$

$$|\alpha, t_0 = 0\rangle = |0\rangle$$

given: $V_{n0} = \langle n|V|0\rangle$
$= v$ (const.)

Find:

a) $c_n^{(1)}(t) = \langle n| U_I^{(1)}(t, t_0)|0\rangle$

b) $|c_n^{(1)}(t)|^2$

c) what $n \neq 0$ has largest $|c_n^{(1)}(t)|^2$

d) asymptotic expression for $|c_n^{(1)}(t)|^2$ for $\Delta t \ll 1$

use $\sin x \approx x \ll 1$

$$\Rightarrow c_n^{(1)}(t) = \langle n | U_I^{(1)}(t, t_0) | i \rangle = \frac{1}{i\hbar} \int_{t_0}^{t} dt\, e^{i(\omega_n - \omega_i)t_1} V_{ni}(t_1)$$

$$= \frac{1}{i\hbar} \int_{t_0}^{t} dt_1\, e^{i(\omega_n - \omega_0)t_1} V_{n0}$$

$$= \frac{V}{i\hbar} \left. \frac{e^{i(\omega_n - \omega_0)t_1}}{i(\omega_n - \omega_0)} \right]_{t_0}^{t}$$

$$c_n^{(1)}(t) = -\frac{V}{\hbar(\omega_n - \omega_0)} \left[e^{i(\omega_n - \omega_0)t} - e^{i(\omega_n - \omega_0)t_0} \right]$$

$$\omega_n - \omega_0 = \frac{E_n - E_0}{\hbar} = \frac{\hbar \Delta n}{\hbar} = \Delta n$$

$$\Rightarrow c_n^{(1)}(t) = -\frac{V}{\hbar \Delta n} \left[e^{i\Delta n t} - e^{i\Delta n t_0} \right]$$

$t_0 = 0$

$$c_n^{(1)}(t) = -\frac{V}{\hbar \Delta n} \left[e^{i\Delta n t} - 1 \right]$$

(b) $$|c_n^{(1)}(t)|^2 = \left(\frac{V}{\hbar \Delta n}\right)^2 \left[e^{i\Delta n t} - 1 \right]\left[e^{-i\Delta n t} - 1 \right]$$

$$= \left(\frac{V}{\hbar \Delta n}\right)^2 \left[2 - e^{i\Delta n t} - e^{-i\Delta n t} \right]$$

$$= \left(\frac{V}{\hbar \Delta n}\right)^2 \left[2(1 - \cos(\Delta n t) \right]$$

$$= \left(\frac{2V}{\hbar \Delta n}\right)^2 \sin^2\left(\frac{\Delta n t}{2}\right)$$

(c) $n = 1$? , not sure why

(d) for asymptotic limit.

$\Delta t << 1$

$\sin x \approx x$

$$= \frac{V^2}{\hbar} t^2$$

* total prob of transition.

$$\sum_{n=1}^{\infty} |c_n^{(1)}(t)|^2 \propto \sum_{n=1}^{\infty} \frac{\sin^2(\Delta n t/2)}{n^2}$$

$$\propto \sum_{n=1}^{\infty} \frac{\sin^2(n \cdot \tilde{t})}{n^2}$$

$\propto t^2$ for small times

* graph of 1 − (total prob.) in Jupyter notebook.

⇒ actually

$$P_{n>0} \approx \underbrace{(\text{const})}_{} t \quad \text{for larger } t$$

$$\equiv \Gamma t$$

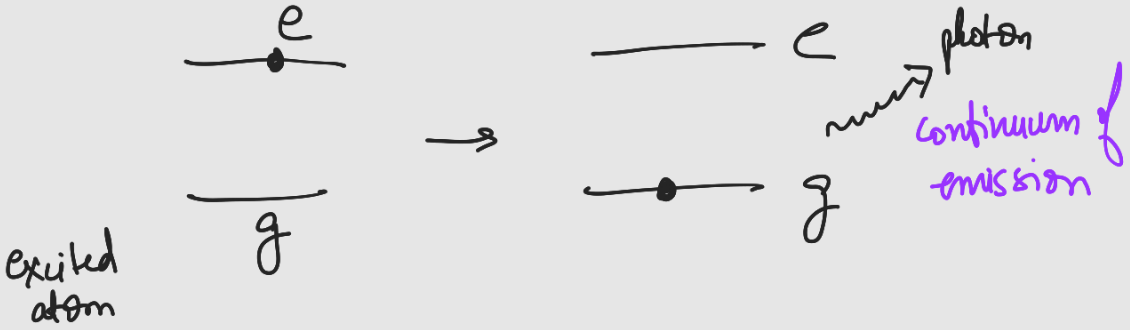
$$= w t$$

↑ transition rate

Continuum of final states

(dense collection of E_{final})

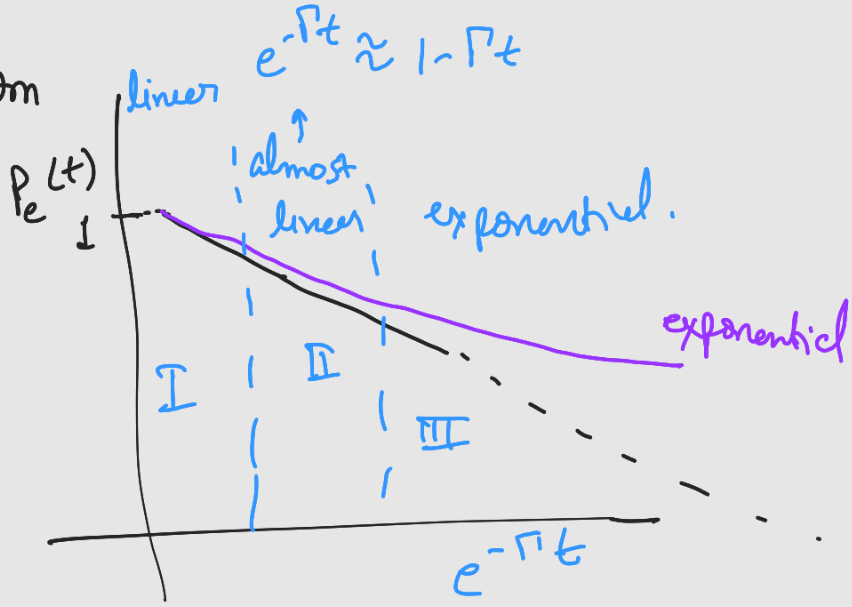
- Example: Spontaneous emission



Continuum: $E_f = \hbar\omega$

Expect: $E_f = \Delta E$

excited atom

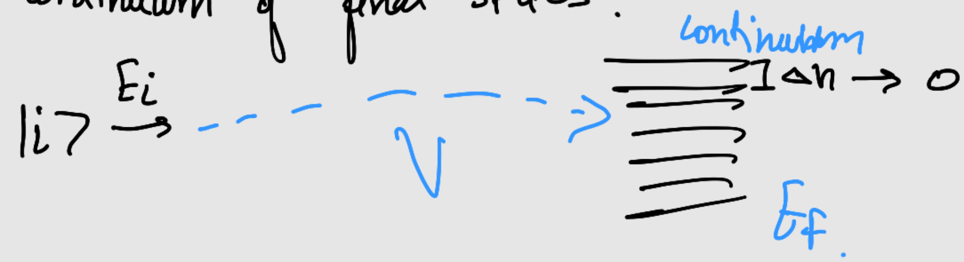


Fermi's Golden Rule:

- constant perturbation for $t \geq 0$

$$H(t) = H_0 + V \quad \text{for } t \geq 0$$

- continuum of final states.



$|f\rangle$

↳ label for final states.

e.g. $|i\rangle = |nlm\rangle$, $|f\rangle = |n'l'm', \vec{k}\,\hat{\varepsilon}\rangle$

wave f^n of photon

$\vec{k} \rightarrow$ wave vector of photon

$\hat{\varepsilon} \rightarrow$ polarisation vector of photon.

$$E_i = E_n \quad , \; E_f = E_{n'} + \hbar c|\vec{k}|$$

- model continuum as discrete energies w/ small spacing.
- Normalized : $\langle f|f'\rangle = \delta_{f,f'}$
- density of states

$$\rho_f(E) = \frac{\#\text{ states}}{\Delta E}$$

Result from this setup.

Rate of transition into continuum:

$$\Gamma = \frac{2\pi}{\hbar}|V_{fi}|^2 \rho_f(E_f)\Big|_{E_f = E_i}$$

$$V_{fi} = \langle f|V|i\rangle \quad \text{for } |f\rangle$$

with $E_f = E_i$

* from warmup, we can conclude that there is a transition factor to continuum.