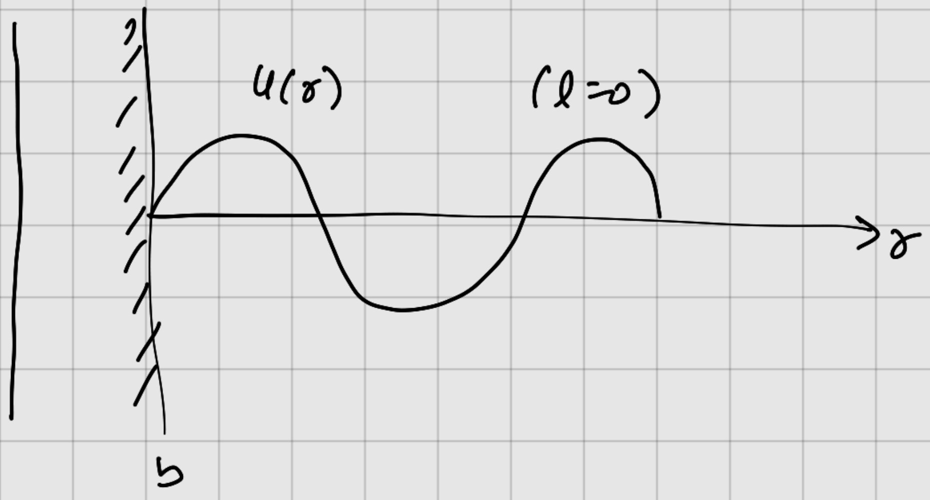
Today: Low-energy scattering.
Sakurai 6.4.4, 6.6-6.6.2

WARM-UP: Hard-sphere Scattering.

$$V(r) = \begin{cases} \infty, & r < b \\ 0, & r > b \end{cases}$$



Radial equation

$$-\frac{\hbar^2}{2m} u''(r) + \left[\frac{l(l+1)}{2mr^2} + V(r)\right] u(r) = \frac{\hbar^2 k^2}{2m} u(r)$$

General solution for $l=0$:

$$u(r) = \begin{cases} 0 & r < b \\ A\cos(kr) + B\sin(kr), & r > b \end{cases}$$

a) apply continuity at $r=b$ to find A/B
b) find $\tan\delta_0$ using $u(r) \to \sin\left(kr - \frac{l\pi}{2} + \delta_l\right)$

$$= \sin(kr + \delta_0)$$

$$= \sin\delta_0 \cos(kr) + \cos\delta_0 \sin(kr)$$

$\Rightarrow$ a) $u(r \to b^-) = u(r \to b^+)$

$$0 = A\cos(kr^+) + B\sin(kr^+)$$

$$\Rightarrow -\tan(kb^+) = \frac{A}{B}$$

0

$$\Rightarrow \frac{A}{B} = -\tan kb$$

b) $\delta_0 = \tan^{-1}\left(-\tan(kb)\right)$

$$= \tan^{-1}\left(\frac{A}{B}\right)$$

$$\boxed{\tan\delta_0 = \frac{A}{B}}$$

—x—

problem (c) find f_0 : $f(\theta) = \sum_{l=0}^{\infty} f_l(\theta)$

$$f_l(\theta) = \frac{2l+1}{k} e^{i\delta_l} \sin\delta_l P_l(\cos\theta)$$

$$\boxed{e^{i\delta}\sin\delta = \frac{\sin\delta}{e^{-i\delta}} = \frac{\sin\delta}{\cos\delta - i\sin\delta} = \frac{\tan\delta}{1-i\tan\delta} = \frac{1}{\cot\delta - i}}$$

$$P_0 = 1$$

Sol.) c)

$$f_0(\theta) = \frac{1}{k}\,\frac{\tan\delta_0}{1-i\tan\delta_0}\cdot P_0$$

$$\tan\delta_0 = A/B$$

$$= \frac{1}{k}\,\frac{A/B}{1-i\,A/B}\cdot 1$$

$$f_0(\theta) = \frac{A}{k(B-iA)}$$

Partial wave cross sec.:

Problem d) Find σ_0, $\sigma = \sum_{l=0}^{\infty} \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l = \sum_{l=0}^{\infty}\sigma_l$

we had, $e^{i\delta}\sin\delta_l = \frac{1}{\cot\delta_l - i}$

complex conjugate : $e^{-i\delta}\sin\delta = \dfrac{1}{\cot\delta_\ell + i}$

multiply these eqn's $\to$ $\sin^2\delta_\ell = \dfrac{1}{\cot^2\delta_\ell + 1}$

$\sigma_0 \Rightarrow l=0$

$\therefore \dfrac{4\pi}{k^2}(2(0)+1)\cdot\dfrac{1}{\cot^2\delta_0+1}$

$= \dfrac{4\pi}{k^2}\cdot\dfrac{1}{\frac{B^2}{A^2}+1}$

$\sigma_0 = \dfrac{4\pi}{k^2}\cdot\dfrac{A^2}{A^2+B^2}$

* some comments:

low-energy limit: $k\to 0$

$l^{max} \cong kb \to 0$

in that case $\sigma \approx \sigma_0$

$kb \ll 1$

$\tan(kb) \approx kb$

$\sigma_0 = \dfrac{4\pi}{k^2\left(\frac{1}{kb}\right)^2 + k^2}$

$= \dfrac{4\pi}{1/b^2 + k^2}$

$$= \frac{4\pi b^2}{1+\underbrace{(kb)^2}_{\text{small}}} \simeq \boxed{4\pi b^2}$$

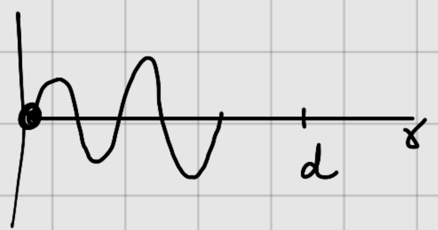
also the surface area.

partial wave cross section in low energy limit.

General method (when $\ell \neq 0$)

suppose $V(r) \approx 0$ for $r > d$

Idea: Solve for $u_\ell(r) = r R_\ell(r)$ for $r < d$



i.e. $V(r) = V_0$ (const.)

$\rightarrow R_\ell(r) = j_\ell(Q r)$

$$E = V_0 + \frac{\hbar^2 Q^2}{2m}$$

for $r > d$, we are assuming $V(r) = 0$

$$R_\ell(r) = a_\ell j_\ell(kr) + b_\ell n_\ell(kr)$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$R_\ell(r) \xrightarrow{r\to\infty} a_\ell \frac{\sin\left(kr - \frac{\ell\pi}{2}\right)}{kr} - b_\ell \frac{\cos\left(kr - \frac{\ell\pi}{2}\right)}{kr}$$

PHASE SHIFT : $R_\ell(r) \rightarrow \frac{c_\ell}{kr} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)$

using

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$= \frac{c_\ell}{kr}\left[\sin\left(kr - \frac{\ell\pi}{2}\right)\cos\delta_\ell + \cos\left(kr - \frac{\ell\pi}{2}\right)\sin\delta_\ell\right]$$

$$\begin{cases} a_\ell = c_\ell \cos\delta_\ell \\ b_\ell = -c_\ell \sin\delta_\ell \end{cases}$$

$$R_\ell(r) \xrightarrow{r\to\infty} a_\ell\left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)\right]$$