Nov-13

Today: Scattering amplitude & Partial wave scattering

WARM-UP

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = E\,u(r) = \frac{\hbar^2 k^2}{2m}u(r)$$

(a) Find a general solution (no B.C.)

(b) Boundary condition: $u(0)=0$ (arbitrary normalization)

$\Rightarrow$ (a) $$\frac{d^2u}{dr^2} = -\frac{\cancel{2m}}{\cancel{\hbar^2}}\frac{\cancel{\hbar^2}k^2}{\cancel{2m}}u(r)$$

$$\Rightarrow \frac{d^2u}{dr^2} = -k^2u(r)$$

$$\boxed{u(r) = A\sin kr + B\cos kr}$$

(b) $u(0) = 0 + B = 0$

$\Rightarrow B=0$

$$\Rightarrow \boxed{u(r) = A\sin kr}$$

- scattering wavefunction

$$\boxed{\psi^{(+)}(\vec{r}) \xrightarrow{r\to\infty} \frac{1}{L^{3/2}}\left(e^{i\vec{k}\cdot\vec{r}} + f(k\hat{r},\vec{k})\frac{e^{ikr}}{r}\right)}$$

for probability conservation

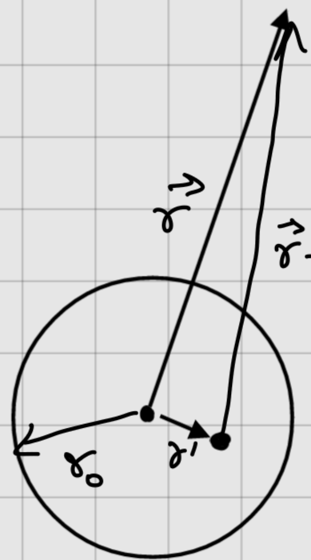
- continuing the derivation from the last time, we had

$$\psi^{(+)}(\vec{r}) = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}} + \frac{2m}{\hbar^2}\int d^3\vec{r}\,' G_+(\vec{r},\vec{r}\,')V(\vec{r}\,')\psi^{(+)}(\vec{r}\,')$$

$$G_+(\vec{r},\vec{r}\,') = -\frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

<u>Large $\vec{r}$</u>

- $|\vec{r}\,'| \lesssim$ Range of $V(r) = r_0$



$r\to\infty$: $r' \ll r$

$$|\vec{r}-\vec{r}\,'| = \sqrt{(\vec{r}-\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}$$

$$= \sqrt{r^2+(\vec{r}\,')^2 - 2rr'\cos\alpha}$$

$$= r\sqrt{1-2\frac{r'}{r}\cos\alpha+\left(\frac{r'}{r}\right)^2}$$

taylor Expansion

$$\approx r\left(1-\frac{r'}{r}\cos\alpha + \underbrace{O\left(\frac{r'}{r}\right)^2}\right)$$

don't need higher order terms as they are negligible.

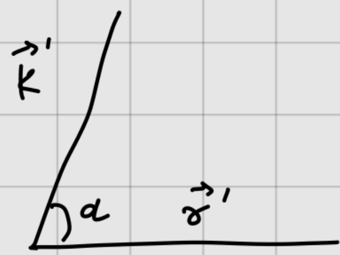
$$= r - r' \cos\alpha$$

$$\therefore \quad e^{ik|\vec{r}-\vec{r}'|} \approx e^{ikr - ikr'\cos\alpha}$$

$$\vec{k}' = k\hat{r}$$

$$k r' \cos\alpha = \vec{k}' \cdot \vec{r}'$$

$$e^{ikr - i\vec{k}' \cdot \vec{r}'}$$



$\therefore$ the Green's f^n is:

$$G_+(\vec{r},\vec{r}') \xrightarrow{r > r'} -\frac{1}{4\pi} e^{ikr} e^{-i\vec{k}'\cdot\vec{r}'} , \quad \text{where } \vec{k}' = k\hat{r}$$

$$\Rightarrow \psi^{(+)}(\vec{r}) \longrightarrow \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}} + \frac{2m}{\hbar^2} \int d^3\vec{r}' \left(\frac{e^{ikr}}{r}\right) e^{-i\vec{k}'\cdot\vec{r}'} V(\vec{r}')\psi^{(+)}(\vec{r}')$$

$$= \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}} - \frac{m L^{3/2}}{2\pi\hbar^2} \left[\int d^3\vec{r}' \, \underbrace{\frac{e^{-i\vec{k}'\cdot\vec{r}'}}{L^{3/2}}}_{\text{to make the } f^n \text{ look like wave } f^n} V(\vec{r}')\psi^{(+)}(\vec{r}')\right] \frac{e^{ikr}}{r}$$

$$= \frac{1}{L^{3/2}} \left[e^{i\vec{k}\cdot\vec{r}} - \frac{mL^3}{2\pi\hbar^2} \left(\int d^3 r' \, \frac{e^{-i\vec{k}'\cdot\vec{r}'}}{L^{3/2}} V(\vec{r}')\psi^{(+)}(\vec{r}') \right) \frac{e^{ikr}}{r} \right]$$

Scat. Ampl.

$$f(\vec{k}', \vec{k}) = -\frac{mL^3}{2\pi\hbar^2} \langle \vec{k}' | V | \underbrace{\psi^{(+)}}_{T|\vec{k}\rangle} \rangle$$

$\hookrightarrow$ initial state

$$= -\frac{mL^3}{2\pi\hbar^2} T_{\vec{k}'\vec{k}}$$

also, we had, the scattering cross-section in terms of T-matrix:

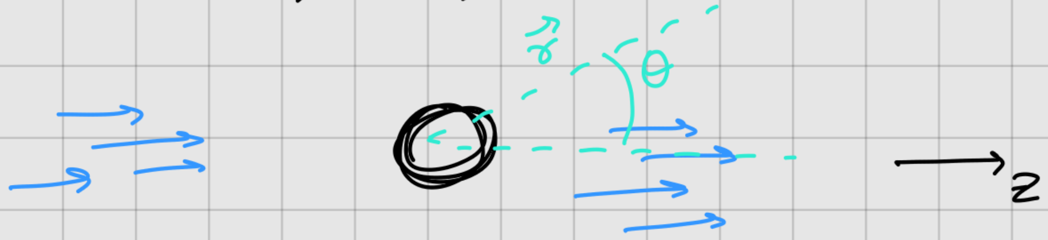
$$\frac{d\sigma}{d\Omega} = \left(\frac{mL^3}{2\pi\hbar^2}\right)^2 |T_{\vec{k},\vec{k}'}|^2$$

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\vec{k}', \vec{k})|^2}$$

Partial Wave analysis

assuming spherical symmetry

- $V(\vec{r}) = V(r)$



* incident particles "breaks" the spherical symmetry to cylindrical symmetry

$\vec{k} = k\hat{z}$ → cylindrical symmetry

- incident: $\varphi_{in}(\vec{r}) = \frac{1}{L^{3/2}} e^{ikz}$

$$\Rightarrow \psi^{(+)}(\vec{r}) = \psi^{(+)}(r, \theta)$$

↳ no ϕ dependence on scattered wave.

∴ all $m = 0$ for for $\psi^{(+)}$ in spherical harmonics representation

$$\therefore \psi^{(+)}(r,\theta) = \sum_{l=0}^{\infty} a_l R_l(r) P_l(\cos\theta)$$

legendre polynomial.

Radial Equation:

- can still write radial eqn even in cylindrical symmetry, by writing sph. eqn for each l

$$(H_0 + V)\psi^{(+)} = E\psi^{(+)}$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}(rR_l) + \left[\frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right](rR_l) = E(rR_l)$$

$$u_l(r) = rR_l(r)$$

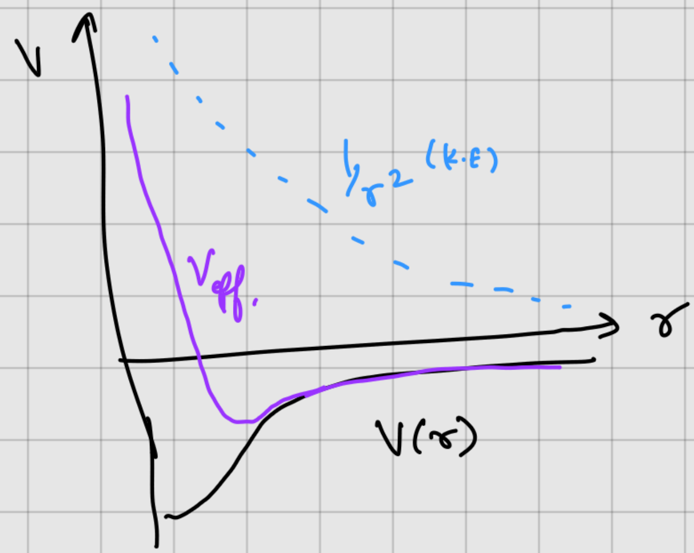
* Effective Potential:

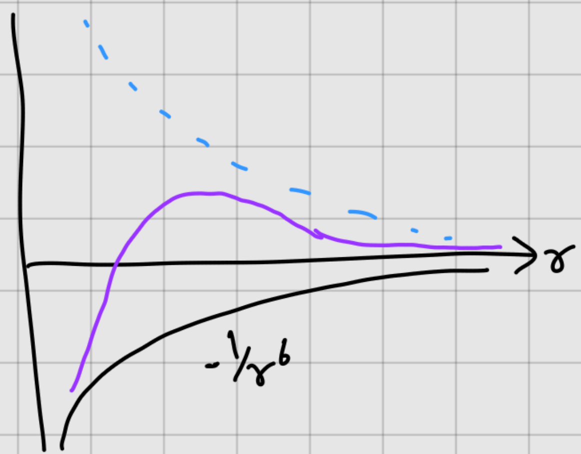
$$V_{eff}(r) = \underbrace{\frac{\hbar^2 l(l+1)}{2mr^2}}_{\hookrightarrow \frac{L^2}{2I} = KE} + V(r)$$

(classical mech. analog)

Boundary condition : $u_l(0) = 0$

Examples

1) 

2) 

Asymptotic Behaviour of $R_\ell(r)$

as $r \to \infty$, $V(r) \to 0$ faster than $\frac{1}{r^2}$ (meaning KE is irrelevant at large r)

$R_\ell(r) \to$ free particle solution.

$$A_\ell \underbrace{j_\ell(kr)}_{\text{spherical Bessel } f^n} + B_\ell \underbrace{n_\ell(kr)}_{\text{sph. Neumann } f^n}$$

$$\underbrace{\frac{1}{kr}\sin\left(kr - \frac{\pi\ell}{2}\right) \qquad \frac{-1}{kr}\cos\left(kr - \frac{\pi\ell}{2}\right)}_{\text{as } r \to \infty}$$

If $V(r) = 0$ everywhere, $rR_\ell = 0$ @ $r = 0$

$$R_\ell = j_\ell(kr) \to \frac{1}{kr}\sin\left(kr - \frac{\ell\pi}{2}\right) \text{ as } r \to \infty$$

General solutions has:

$$R_\ell(r) \to \frac{1}{kr}\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right), \text{ for some } \delta_\ell$$

phase shift

for $l=0$

$\delta_0 \to$ s-wave phase shift $\to$ only algebraically easily derived

$l=1$, $\delta_1 \to$ p-wave phase shift

$l=2$, $\delta_2 \to$ d-wave phase shift

higher l can be calculated computationally.