Nov 11

Scattering Amplitude (6.2)

WARM UP: Periodic B.C. Box

$$\langle \vec{r} | \vec{k} \rangle = \frac{1}{L^{3/2}} e^{i \vec{k}\cdot\vec{r}}$$

$$\vec{k} = \frac{2\pi}{L} (n_x, n_y, n_z)$$

Find

$$\int d^3\vec{k}\, f(\vec{k}) \simeq (?) \sum_{\vec{k}} f(\vec{k})$$

HINT

$$\sum_x \Delta x \, f(x) \approx \int dx \, f(x)$$

as $\delta x \to 0$

have to find this

$$\Delta k = \frac{2\pi}{L}$$

$$(\Delta k)^3 = \left(\frac{2\pi}{L}\right)^3 \quad (\because \text{ the integral is 3-D})$$

$$\Rightarrow \quad ? \rightarrow \left(\frac{2\pi}{L}\right)^3$$

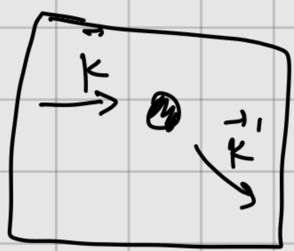
$$\therefore \int d^3k \, f(k) = \left(\frac{2\pi}{L}\right)^3 \sum_k f(k)$$

Review:

initial $|i\rangle$: $\langle \vec{r}|i\rangle = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}}$, $E = \frac{\hbar^2 k^2}{2m}$

Potential : $V(\vec{r})$

final $|f\rangle$: $\langle \vec{r}|f\rangle = \frac{1}{L^{3/2}} e^{i\vec{k}'\cdot\vec{r}}$



* we don't know what $\vec{k}'$ is (b/c Quantum Mech ☺), but we can predict it's probability distribution.

Differential Cross-Section

$$\frac{d\sigma}{d\Omega} = \left(\frac{mL^3}{2\pi\hbar^2}\right)^2 |T_{fi}|^2$$

where : $T = V + VG_0T$

describes propagation (how?)

it's called resolvent in this form

$$G_0(E) = \frac{1}{E - H_0 + i\eta} \quad (\eta \to 0^+)$$

$$H_0 = \frac{\vec{p}^{\,2}}{2m} = -\frac{\hbar^2}{2m}\nabla^2$$

(operator)

Scattering states

$$T_{fi} = \langle f | T | i \rangle = \langle f | V | \psi^{(+)} \rangle$$

Find $|\psi^{(+)}\rangle$, satisfies

- $(H_0 + V)|\psi^{(+)}\rangle = E|\psi^{(+)}\rangle$, $E \geq 0$
- L.S.E

$$\bullet\ \psi^{(+)}(\vec{r}) = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}} + \frac{2m}{\hbar^2}\int d^3\vec{r}\,'\, G_+(\vec{r},\vec{r}\,')\, V(\vec{r}\,')\,\psi^{(+)}(\vec{r}\,')$$

$$G_+(\vec{r},\vec{r}\,') = \frac{\hbar^2}{2m}\langle \vec{r} | \frac{1}{E - H_0 + i\eta} | \vec{r}\,' \rangle$$

$$\mathbb{1} = \sum_{\vec{k}'} |\vec{k}'\rangle\langle\vec{k}'|$$

Green's function

$$G_+(\vec{r},\vec{r}\,') = \frac{\hbar^2}{2m}\langle \vec{r} | \frac{1}{E - H_0 + i\eta} \sum |\vec{k}'\rangle\langle\vec{k}'| \vec{r}\,' \rangle$$

projectors.

$$= \frac{\hbar^2}{2m} \sum_{\vec{k}'} \langle \vec{r} \Big| \frac{1}{\left(\frac{\hbar^2 k^2}{2m}\right) - \left(\frac{\hbar^2 (k')^2}{2m}\right) + i\eta} \Big| \vec{k}' \rangle \frac{1}{L^{3/2}} e^{-i\vec{k}'\cdot\vec{r}'}$$

$$= \underbrace{\sum_{\vec{k}'} \frac{1}{L^3}}_{\frac{1}{(2\pi)^3}} e^{i\vec{k}'\cdot\vec{r}} \frac{1}{k^2 - (k')^2 + i\varepsilon} e^{-i\vec{k}'\cdot\vec{r}'}$$

(from warm-up)

$$= \frac{1}{(2\pi)^3} \int d^3\vec{k}' \frac{e^{i\vec{k}'(\vec{r}-\vec{r}')}}{k^2 - (k')^2 + i\varepsilon}$$

$$G_+(\vec{r}, \vec{r}') = \int \frac{d^3 k'}{(2\pi)^3} \frac{e^{i\vec{k}'\cdot(\vec{r}-\vec{r}')}}{k^2 - (k')^2 + i\varepsilon}$$

$$k^2 - (k')^2 + i\varepsilon = -(k'-a)(k'-b)$$ (factorizing)

$$= -(k')^2 + (b+a)k' - ab$$

by comparison $\Rightarrow$ $\underbrace{a+b=0, \quad ab = i\varepsilon}_{}$

$$a = \sqrt{k^2+i\varepsilon}, \quad b = -\sqrt{k^2+i\varepsilon}$$

$$\Rightarrow a = k\sqrt{1+i\frac{\varepsilon}{k^2}}, \quad b = -k\sqrt{1+i\frac{\varepsilon}{k^2}}$$

Since ε is small,
taylor expanding the R.H.S

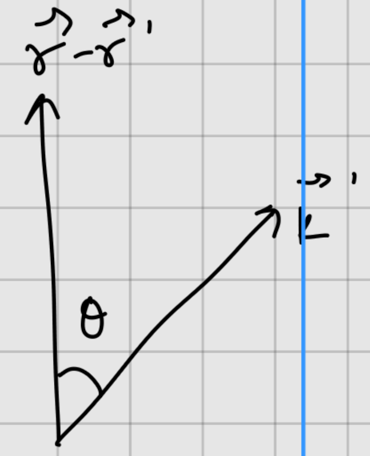
$\Rightarrow$ $\boxed{a \approx k + \frac{i\varepsilon}{2k^2}}$ $\boxed{b \approx -k - \frac{i\varepsilon}{2k^2}}$

$a = k + i\delta$ $\quad a = -k - i\delta$ (say)

$$\therefore \quad G_+(\vec{r}, \vec{r}\,') = -\int \frac{d^3k'}{(2\pi)^3} \frac{e^{i\vec{k}'\cdot(\vec{r} - \vec{r}\,')}}{(k' - (k + i\delta))(k' + (k - i\delta))}$$

⋮

* let z axis be along $\vec{r} - \vec{r}\,'$



⋮

$$= \int_0^\infty \frac{(k')^2 dk'}{(2\pi)^3} \int_0^\pi d\theta \sin\theta \, 2\pi \frac{e^{ik'|\vec{r} - \vec{r}\,'|\cos\theta}}{(k' - (k + i\delta))(k' + (k - i\delta))} (-1)$$

$$= \int \frac{(k')^2 dk'}{(2\pi)^3} 2\pi \int_{-1}^{1} du \frac{e^{ik|\vec{r} - \vec{r}\,'|u}}{[k' - a][k' + a]}$$

$$= \int_0^\infty \frac{(k')^2 dk'}{(2\pi)^2} \frac{e^{ik'|\vec{r}-\vec{r}'|} - e^{-ik'|\vec{r}-\vec{r}'|}}{(ik')(|\vec{r}-\vec{r}'|)(k'-a)(k'+a)} (-1)$$

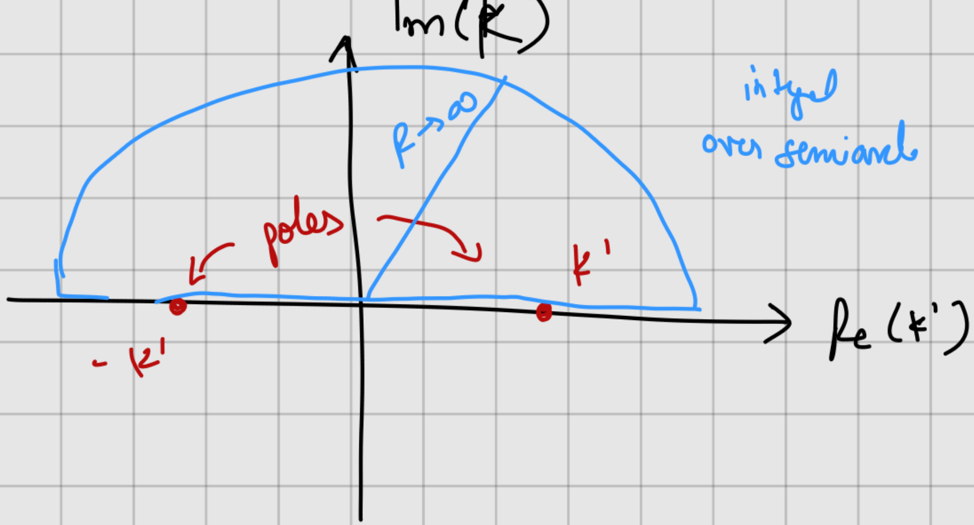
$$G_+(\vec{r},\vec{r}') = \frac{i}{|\vec{r}-\vec{r}'|} \int \frac{k' dk'}{(2\pi)^2} \frac{e^{ik'|\vec{r}-\vec{r}'|}}{(k'-(k+i\delta))(k'+(k-i\delta))}$$

⋮

Using Cauchy residue theorem

$$\oint \frac{f(z)}{z-z_0} dz = 2\pi i \, f(z_0)$$

$\hookrightarrow k'$



$$G_+(\vec{r},\vec{r}') = \frac{i}{|\vec{r}-\vec{r}'|} \oint \frac{z\,dz}{(2\pi)^2} \frac{e^{iz|\vec{r}-\vec{r}'|}}{(z-(k+i\delta))(z-(-k-i\delta))}$$

$$= \frac{i}{|\vec{r}-\vec{r}'|} (2\pi i) \frac{(k+i\delta)}{(2\pi)^2} \frac{e^{i(k+i\delta)|\vec{r}-\vec{r}'|}}{(k+i\delta)-(-k-i\delta)}$$

$$\boxed{G_+(\vec{r},\vec{r}\,') = -\frac{1}{4\pi|\vec{r}-\vec{r}\,'|}\, e^{ik|\vec{r}-\vec{r}\,'|}}$$