Dec-2

Today:
- Identical Particles
- Wavefn Symmetry (Zettili 8.2)
- Scattering (Zettili 11.3, Sakurai 6.8)

$S^2|sm\rangle = \hbar^2 s(s+1)|sm\rangle$
$S_z|sm\rangle = m\hbar|sm\rangle$

WARM-UP: Two spin 1/2 particles $\vec{S}_1, \vec{S}_2$

basis: $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

List Eig. states, eigen.vals of $\vec{S}^2, S_z$

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

$$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2S_1\cdot S_2$$

$\uparrow: m = 1/2, s = \hbar/2$

$\downarrow: m = -1/2, s = -\hbar/2$

$$S_z|\overset{|sm\rangle}{\uparrow\uparrow}\rangle = m\hbar|\uparrow\uparrow\rangle = \frac{\hbar}{2}|\uparrow\uparrow\rangle$$

$$S_z|\uparrow\downarrow\rangle = -\hbar/2\,|\uparrow\downarrow\rangle$$

$$S_z|\downarrow\uparrow\rangle = \hbar/2\,|\downarrow\uparrow\rangle$$

$$S_z|\downarrow\downarrow\rangle = -\hbar/2\,|\downarrow\downarrow\rangle$$

other linear combinations of basis kets are also kets. Possible permutations lead to eigenvalues:

two states: $\hbar, 0, -\hbar$

three: $\hbar/2, -\hbar/2$

four: D.

possible eigenvalues of S_z: $-\hbar/2, \hbar/2, \hbar, -\hbar$

Class answer

$S = 0, 1$ (solve later)

$S = 1, \quad M = -1, 0, 1$

"triplet"
even under
$\vec{S}_1 \leftrightarrow \vec{S}_2$

$$\left\{ \begin{aligned} &|S=1, M=1\rangle = |\uparrow\uparrow\rangle \\ &|S=1, M=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ &|S=1, M=-1\rangle = |\downarrow\downarrow\rangle \end{aligned} \right.$$

$S=0$: $M=0$

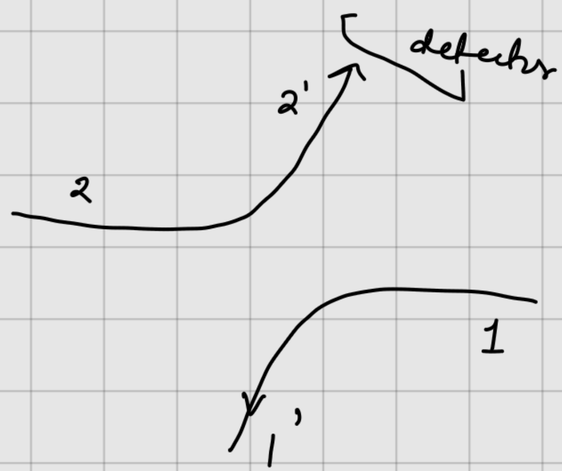
odd under
particle exchange

$$|S=0, M=0\rangle = -\frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle$$

Two-Particle Scattering Review:

- Target: $M_1, \vec{r}_1, \vec{p}_1$
- Detect: $M_2, \vec{r}_2, \vec{p}_2$

} starting off with distinguishable particles.



- Relative Position: $\vec{r} = \vec{r}_2 - \vec{r}_1$
- C.M. Position: $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$

proceeding with classical approach (there's also a quantum approach for the following derivation but classical is easier)

$$K.E. = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2}$$

$$= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 = \mathcal{L}$$

$$M = m_1+m_2\,, \quad \mu = \frac{m_1 m_2}{m_1+m_2}$$

MOMENTA:

total: $\vec{P} = \vec{p}_1 + \vec{p}_2$

Relative momentum:

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = \mu \dot{\vec{r}} = \mu \left(\frac{p_2}{m_2} - \frac{p_1}{m_1} \right)$$

$$= \frac{m_1 \vec{p}_2 - m_2 \vec{p}_1}{m_1 + m_2}$$

which reduces the two body problem to one body problem.

$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|)$$

$$= \underbrace{\frac{\vec{P}^2}{2M}}_{C.M.} + \underbrace{\frac{\vec{p}^2}{2\mu} + V(r)}_{\text{Relative}}$$

- Two-particle Wavefunction $\psi(\vec{r}_1, \vec{r}_2)$
- Seperation of variables : $\psi(\vec{r}_1, \vec{r}_2) = \Phi(\vec{R})\,\Psi(\vec{r})$
- Dirac : $|\psi\rangle_{2p} = |\Phi\rangle_{cm}\,|\Psi\rangle_{rel}$

T.I.S.E : $H|\psi\rangle = E|\psi\rangle$

Suppose : $\frac{\vec{P}^2}{2m} |\Phi\rangle = E_{cm} |\Phi\rangle$

$$\left(\frac{p^2}{2\mu} + V(r) \right) |\Psi\rangle = E_{rel} |\Psi\rangle$$

Check:

$$\left(\frac{P^2}{2m}+\frac{p^2}{2\mu}+V\right)|\Phi\rangle|\psi\rangle = \frac{P^2}{2m}|\Phi\rangle|\psi\rangle$$

$$+ |\phi\rangle\left(\frac{p^2}{2\mu}+V\right)|\psi\rangle$$

$$= \underbrace{(E_{cm}+E_{rel})}_{E}|\Phi\rangle|\psi\rangle$$

Two-particle Scattering

- C.M. frame : $\vec{P}|\Phi\rangle = 0$, $E_{cm} = 0$

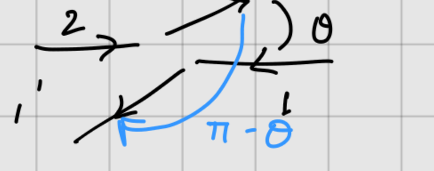
Scattering : Distinguishable

$\sim$ Solve T.I.S.E etc., mass μ

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = |f(\theta)|^2 \quad \text{in C.M. frame}$$

- detect particle 2

Classical Identical particles / indistinguishable



$$\frac{d\sigma_{cl}(\theta)}{d\Omega} = \frac{d\sigma(\theta)}{d\Omega} + \frac{d\sigma(\pi-\theta)}{d\Omega}$$

Classical formula

- assuming

using: $\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_1{}' + \vec{p}_2{}'$ (CM frame)

Quantum Identical Particle Scattering

- Let $f(\theta)$ ignoring identical Nature.

So far

- use $\mu = \frac{m_1 m_2}{m_1+m_2} = \frac{m^2}{m+m} = \frac{m}{2}$

now

account for wavefunction symmetry.

$$f_{\text{symmetric/anti-symmetric}} = f(\theta) \pm f(\pi-\theta)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta) \pm f(\pi-\theta)|^2$$

Quantum formula.

Identical Particles (Zettili Ch. 8)

• Two-particle wavefunction w/ spin.

$$\Psi(1,2) = \Psi(\underbrace{\vec{r}_1, m_{s1}}_{\text{"1"}} ; \underbrace{\vec{r}_2, m_{s2}}_{\text{"2"}})$$

$$= \psi(\vec{r}_1, \vec{r}_2)\, \chi(m_{s1}, m_{s2})$$

Example: $\vec{S} = \vec{S}_1 + \vec{S}_2$

Eigen. state of $\vec{S}^2, S_z$; $|\chi\rangle = |S, M_S\rangle$

$$\chi(m_{s1}, m_{s2}) = \langle m_{s1}, m_{s2} | \chi \rangle$$

$$= \langle m_{s1}, m_{s2} | S\, M_s \rangle$$

$$= \chi_{ms1, ms2}$$

for <u>identical Particles</u>

$$|\Psi(1,2)|^2 = |\Psi(2,1)|^2$$

* Spin-statistics theorem *

Fermions $(s_1 = s_2 = 1/2, 3/2, \ldots)$: $\Psi(1,2)$

$$= -\Psi(2,1)$$

Bosons $(s_1 = s_2 = 0, 1, 2, \ldots)$: $\Psi(1,2) = \Psi(2,1)$

- Assume $\chi(m_{s_1}, m_{s_2}) = \pm \chi(m_{s_2}, m_{s_1})$
- Let χ_s = symmetric f^n (even)
  χ_A = Anti symmetric (odd)

Fermions: $\Psi_A = \varphi_A \chi_s$ or $\varphi_s \chi_A$

Bosons: $\Psi_s = \varphi_s \chi_s$ or $\varphi_A \chi_A$

Scattering

$$\varphi_s(\vec{r}_1, \vec{r}_2) = \Phi(R)\, \psi(\vec{r})$$

ψ → even

$$\psi_{even}(-\vec{r}) = \psi_{even}(\vec{r})$$

$$\varphi_A(\vec{r}_1, \vec{r}_2) = \Phi(\vec{R})\, \psi_{odd}(\vec{r})$$

Scattering amplitude

$$f_s(\theta) = f(\theta) + f(\pi - \theta)$$

$$f_A(\theta) = f(\theta) - f(\pi - \theta)$$

Total spin:

fermions.

e.g. $S=1 \to \chi_S \to \psi_A \to f_A(\theta) = f(\theta) - f(\pi-\theta)$

$(s_1 = s_2 = 1/2)$

$S=0 \to \chi_A \to f_S = f(\theta) + f(\pi-\theta)$